FULL CHAINS OF TWISTS FOR ORTHOGONAL ALGEBRAS

Petr P. Kulish
St.Petersburg Department of the Steklov Mathematical Institute,
191011, St.Petersburg, Russia
Vladimir D. Lyakhovsky
Theoretical Department, St. Petersburg State University,
198904, St. Petersburg, Russia
Alexander A. Stolin
Department of Mathematics, University of Goteborg,
S-412 96 Goteborg, Sweden

Abstract

We show that for some Hopf subalgebras in $U_{\mathcal{F}}(so(M))$ nontrivially deformed by a twist $\mathcal{F}$ it is possible to find the nonlinear primitive copies. This enlarges the possibilities to construct chains of twists. For orthogonal algebra $U(so(M))$ we present a method to compose the full chains with carrier space as large as the Borel subalgebra $B(so(M))$. These chains can be used to construct the new deformed Yangians.

1 Introduction

Quantizations of triangular Lie bialgebras $\mathbf{L}$ with antisymmetric classical r-matrices $r = -r_{21}$ are defined by a twisting element $\mathcal{F} = \sum f_{(1)} \otimes f_{(2)} \in \mathcal{A} \otimes \mathcal{A}$ which satisfies the twist equations [1]:

$$\begin{array}{l} (\mathcal{F})_{12} \, (\Delta \otimes \mathrm{id}) \, \mathcal{F} = (\mathcal{F})_{23} \, (\mathrm{id} \otimes \Delta) \, \mathcal{F}, \\ (\epsilon \otimes \mathrm{id}) \, \mathcal{F} = (\mathrm{id} \otimes \epsilon) \, \mathcal{F} = 1. \end{array} \tag{1}$$

Explicit form of the twisting element is quite important in applications because it provides explicit expressions for the quantum $\mathcal{R}$-matrix $\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21} \mathcal{F}^{-1}$ and for the twisted coproduct $\Delta_{\mathcal{F}} \, () = \mathcal{F} \Delta_{\mathcal{F}} \, () \, \mathcal{F}^{-1}$.

The first nontrivial explicitly written twisting elements $\mathcal{F}$ were given in the papers [2], [3], [4] and [5]. These twists can be defined on the following

carrier algebra L:

$$\begin{array}{cc} [H,E]=E, & [H',E]=\gamma' E, \\ [H,A]=\alpha A, & [H',A]=\alpha' A, \\ [H,B]=\beta B, & [H',B]=\beta' B, \\ [E,A]=[E,B]=0, & [A,B]=E \\ \alpha+\beta=1, & \alpha'+\beta'=\gamma'. \end{array}$$

Explicit expressions for their twisting elements are

$$\begin{array}{ll} \Phi_{\mathcal{R}}=e^{H\otimes H'}, & r_{\mathcal{R}}=H\wedge H', \\ \Phi_{\mathcal{J}}=e^{H\otimes\sigma}, & r_{\mathcal{J}}=H\wedge E, \\ \Phi_{\mathcal{E}\mathcal{J}}=\Phi_{\mathcal{E}}\Phi_{\mathcal{J}}=e^{A\otimes Be^{-\beta\sigma}}e^{H\otimes\sigma}, & r_{\mathcal{E}\mathcal{J}}=H\wedge E+A\wedge B, \\ \sigma=\ln(1+E)\,. & \end{array} \tag{2}$$

Here $r_{\mathcal{R}}, r_{\mathcal{J}}, r_{\mathcal{E}\mathcal{J}}$ are the corresponding classical r-matrices.

Carrier subalgebras $\mathbf{L}$ can be found in any simple Lie algebra g of rank greater than 1.

It was demonstrated in [6] that these twists can be composed into *chains*. They are based on the sequences of regular injections constructed for the initial Lie algebra

$$g_p \subset g_{p-1} \ldots \subset g_1 \subset g_0 = g.$$

To form the chain one must choose an initial root λ_0 in the root system $\Lambda(g)$, consider the set π of its *constituent roots*

$$\begin{array}{l} \pi=\{\lambda',\lambda'' \mid \lambda'+\lambda''=\lambda_0; \quad \lambda'+\lambda_0, \lambda''+\lambda_0 \notin \Lambda(g)\} \\ \pi=\pi'\cup\pi''; \qquad \pi'=\{\lambda'\}\,, \pi''=\{\lambda''\}\,. \end{array}$$

and the subset $\Lambda^{\perp}_{\Lambda_0}$ of roots orthogonal to λ_0 (the corresponding subalgebra in g will be denoted by $g^{\perp}_{\lambda_0}$).

It was shown that for the classical Lie algebras g one can always find in $g^{\perp}_{\lambda_0}$ a subalgebra $g_1 \subseteq g^{\perp}_{\lambda_0} \subset g_0 = g$ whose generators become primitive after the extended twist $\Phi_{\mathcal{E}\mathcal{J}}$. Such primitivization of $g_k \subset g_{k-1}$ (called the *matreshka* effect [6]) provides the possibility to compose chains of extended twists of the type $\Phi_{\mathcal{E}\mathcal{J}}$,

$$\begin{array}{l} \mathcal{F}_{\mathcal{B}_{0\prec p}}=\prod_{k=0}^{p}\Phi_{\mathcal{E}_k}\Phi_{\mathcal{J}_k}, \\ \Phi_{\mathcal{E}_k}\Phi_{\mathcal{J}_k}=\prod_{\lambda'\in\pi'_k}\exp\left\{E_{\lambda'}\otimes E_{\lambda_0^k-\lambda'}e^{-\frac{1}{2}\sigma_{\lambda_0^k}}\right\}\cdot\exp\{H_{\lambda_0^k}\otimes\sigma_{\lambda_0^k}\}\,. \end{array} \tag{3}$$

Chains of twists quantize a large variety of r-matrices corresponding to Frobenius subalgebras in simple Lie algebras [7].

2 Construction of a full chain of twists

The main point in the construction of a chain is the *invariance* of g_{k+1} with respect to $\Phi_{\mathcal{E}_k \mathcal{J}_k}$. When these subalgebras are proper the canonical chains have only a part of $B^+(g)$ as the twist carrier subalgebra:

$$\begin{array}{ccccccccc} \ldots & \subset g^{\perp}_{\lambda_0^k} & \subset g^{\perp}_{\lambda_0^{k-1}} & \ldots & \subset g^{\perp}_{\lambda_0^1} & \subset g^{\perp}_{\lambda_0^0} & \subset g \\ & \cap & \cap & & \cap & \cap & \| \\ \ldots & \subset g_{k+1} & \subset g_k & \ldots & \subset g_2 & \subset g_1 & \subset g_0 \end{array} \tag{4}$$

We would like to demonstrate that the effect of primitivization is *universal* and extends to the whole subalgebra $g^{\perp}_{\lambda_0^k}$. It was shown in [8] that the invariance of a subalgebra in $g^{\perp}_{\lambda_0^k}$ is only one of the forms of the *primitivization* . In general this is the existence (in the twisted Hopf algebra $U_{\mathcal{E}_k \mathcal{J}_k}\left(g^{\perp}_{\lambda_0^k}\right)$) of a primitive subspace V_G^{k+1} with the algebraic structure isomorphic to $g^{\perp}_{\lambda_0^k}$. On this subspace the subalgebra $g^{\perp}_{\lambda_0^k}$ is realized nonlinearly so V_G^{k+1} is called *deformed carrier space* [8].

In this context the situation with the twists for $U(sl(N))$ is degenerate: the subalgebra $(sl(N))^{\perp}_{\lambda_0^k}$ coincides with $(sl(N))_{k+1}$, i.e. $V_G^{k+1} = V_{(sl(N))^{\perp}_{\lambda_0^k}}$.

In the case of $U(so(M))$ the situation is different. Let the root system $\Lambda(so(M))$ be

$$\{\pm e_i \pm e_j \mid i,j = 1,2,\ldots M/2; i \neq j\}$$

for even M and

$$\{\pm e_i \pm e_j; \pm e_k \mid i,j,k = 1,2,\ldots (M-1)/2; i \neq j\}$$

for odd M. Take $e_1 + e_2$ as the initial root. Here the subalgebras $g^{\perp}_{\lambda_0^{k-1}}$ and g_k in (4) are related as follows,

$$g^{\perp}_{\lambda_0^{k-1}} = g_k \oplus so^{(k)}(3) = so(M-4k) \oplus so^{(k)}(3).$$

Consider the invariants of the vector fundamental representations of $g_{k+1} = so(M - 4(k+1))$ acting on g_k :

$$\begin{array}{l} I^a_{2N+1} = \frac{1}{2}E_a^2 + \sum_{l=3}^N (E_{a+l}E_{a-l}), \\ I^{a\otimes b}_{2N+1} = E_a \otimes E_b + \sum_{l=3}^N (E_{a+l}\otimes E_{b-l} + E_{a-l}\otimes E_{b+l}), \end{array} \tag{5}$$

$$
\begin{array}{l}
I_{2N}^{a}=\sum_{l=3}^{N}\left(E_{a+l}E_{a-l}\right), \\
I_{2N}^{a\otimes b}=\sum_{l=3}^{N}\left(E_{a+l}\otimes E_{b-l}+E_{a-l}\otimes E_{b+l}\right),
\end{array}
\qquad (6)
$$

The $so^{(k)}(3)$ summands are non-trivially deformed by $\Phi_{\mathcal{E}_{k-1}\mathcal{J}_{k-1}}$:

$$
\begin{aligned}
\Delta_{\mathcal{E}_{k-1}\mathcal{J}_{k-1}}\left(E_{1-2}^{k}\right) = {} & E_{1-2}^{k}\otimes 1+1\otimes E_{1-2}^{k}+\left(1\otimes e^{-\frac{1}{2}\sigma_{1+2}^{k-1}}\right)I_{M-4k}^{1\otimes 1} \\
& +I_{M-4k}^{1}\otimes\left(e^{-\sigma_{1+2}^{k-1}}-1\right), \\
\Delta_{\mathcal{E}_{k-1}\mathcal{J}_{k-1}}\left(E_{2-1}^{k}\right) = {} & E_{2-1}^{k}\otimes 1+1\otimes E_{2-1}^{k}+\left(e^{\sigma_{1+2}^{k-1}}-1\right)\otimes I_{M-4k}^{2}e^{-\sigma_{1+2}^{k-1}} \\
& +\left(1\otimes e^{-\frac{1}{2}\sigma_{1+2}^{k-1}}\right)I_{M-4k}^{2\otimes 2}.
\end{aligned}
$$

According to the main principle formulated above (despite the deformed costructure of $V_{g^{\perp}_{\lambda_0^{k-1}}}$) the primitivization is realized on its isomorphic image V_G^{k+1} contained in $U_{\mathcal{E}_{k-1}\mathcal{J}_{k-1}}\left(g^{\perp}_{\lambda_0^{k-1}}\right)$. To find this deformed carrier subspace V_G^{k+1} it is sufficient to inspect the coproducts of invariants (5) and (6),

$$
\begin{aligned}
& \Delta_{\mathcal{E}_{k}\mathcal{J}_{k}}\left(I_{M-4k}^{1}\right)= \\
& =I_{M-4k}^{1}\otimes e^{-\sigma_{1+2}^{k}}+1\otimes I_{M-4k}^{1}+I_{M-4k}^{1\otimes 1}\left(1\otimes e^{-\frac{1}{2}\sigma_{1+2}^{k}}\right),
\end{aligned}
$$

$$
\begin{aligned}
& \Delta_{\mathcal{E}_{k}\mathcal{J}_{k}}\left(I_{M-4k}^{2}e^{-\sigma_{1+2}^{k}}\right)= \\
& =I_{M-4k}^{2}e^{-\sigma_{1+2}^{k}}\otimes 1+e^{\sigma_{1+2}^{k}}\otimes I_{M-4k}^{2}e^{-\sigma_{1+2}^{k}}+I_{M-4k}^{2\otimes 2}\left(1\otimes e^{-\frac{1}{2}\sigma_{1+2}^{k}}\right).
\end{aligned}
$$

Now one can construct the following nonlinear primitive generators

$$
\begin{array}{ll}
G_{1-2}^{k+1}=E_{1-2}^{k}-I_{M-4k}^{1}, & \Delta_{\mathcal{E}_{k}\mathcal{J}_{k}}\left(G_{1-2}^{k+1}\right)=G_{1-2}^{k+1}\otimes 1+1\otimes G_{1-2}^{k+1}, \\
G_{2-1}^{k+1}=E_{2-1}^{k}-I_{M-4k}^{2}e^{-\sigma_{1+2}}, & \Delta_{\mathcal{E}_{k}\mathcal{J}_{k}}\left(G_{2-1}^{k+1}\right)=G_{2-1}^{k+1}\otimes 1+1\otimes G_{2-1}^{k+1}, \\
H_{1-2}^{k-1}, & \Delta_{\mathcal{E}_{k}\mathcal{J}_{k}}\left(H_{1-2}^{k-1}\right)=H_{1-2}^{k-1}\otimes 1+1\otimes H_{1-2}^{k-1}.
\end{array}
$$

The subspace spanned by $\left\{H_{1-2}^{k},G_{1-2}^{k+1},G_{-1+2}^{k+1}\right\}$ forms the algebra $so_G^{(k+1)}(3)\approx so^{(k+1)}(3)$:

$$
\begin{aligned}
\left[H_{1-2}^{k},G_{1-2}^{k+1}\right] &= G_{1-2}^{k+1}, \\
\left[H_{1-2}^{k},G_{2-1}^{k+1}\right] &= -G_{2-1}^{k+1}, \\
\left[G_{1-2}^{k+1},G_{2-1}^{k+1}\right] &= 2H_{1-2}^{k}.
\end{aligned}
$$

Therefore we obtain the deformed primitive space

$$
V_G^{k+1}\left(g^{\perp}_{\lambda_0^{k}}\right)=V\left(g_{k+1}\right)\oplus V\left(so_G^{(k+1)}(3)\right),
$$

that can be considered as a carrier for the twists (2). The next extended Jordanian twist in the chain (that is defined on g_{k+1}) does not touch the space $V\left(so_G^{(k+1)}(3)\right)$. Consequently after all the steps of the chain we will still have a primitive subalgebra

$$\mathcal{D}=\sum_{k=0}^{p}{}^{\oplus}so_G^{(k+1)}(3)$$

defined on the sum of deformed spaces $V\left(so_G^{(k+1)}(3)\right)$.

Thus in the twisted Hopf algebra $U_{\mathcal{B}_{0\prec p}}\left(so\left(M\right)\right)$ one can perform further twist deformations with the carrier subalgebra in $\mathcal{D}$. The most interesting among them are the Jordanian twists defined by

$$\Phi_{\mathcal{J}_k}^G=\exp\left(H_{1-2}^k\otimes\sigma_G^k\right)\quad\text{with}\qquad\sigma_G^k\equiv\ln\left(1+G_{1-2}^{k+1}\right)$$

This means that in the general expression for the twisting element $\mathcal{F}_{\mathcal{B}_{0\prec p}}$ one can insert in the appropriate $k\geq 0$ places the Jordanian twisting factors defined on the deformed carrier spaces, i.e. to perform a substitution

$$\begin{array}{l}\Phi_{\mathcal{E}_k}\Phi_{\mathcal{J}_k}\Rightarrow\Phi_{\mathcal{J}_k}^G\Phi_{\mathcal{E}_k}\Phi_{\mathcal{J}_k}\equiv\Phi_{\mathcal{G}_k}\\ \exp\left\{I_{M-4k}^{1\otimes 2}\left(1\otimes e^{-\frac{1}{2}\sigma_{1+2}^k}\right)\right\}\cdot\exp\{H_{1+2}^k\otimes\sigma_{1+2}^k\}\Rightarrow\\ \exp\left(H_{1-2}^k\otimes\sigma_G^k\right)\cdot\exp\left\{I_{M-4k}^{1\otimes 2}\left(1\otimes e^{-\frac{1}{2}\sigma_{1+2}^k}\right)\right\}\cdot\exp\left(H_{1+2}^k\otimes\sigma_{1+2}^k\right)\end{array}$$

This gives *the full chain* in the following form

$$\begin{array}{l}\mathcal{F}_{\mathcal{G}_{0\prec p}}=\prod_{k=p}^{0}\Phi_{\mathcal{G}_k}=\\ \prod_{k=p}^{0}\left(\exp\left(H_{1-2}^k\otimes\sigma_G^k\right)\cdot\exp\left\{I_{M-4k}^{1\otimes 2}\left(1\otimes e^{-\frac{1}{2}\sigma_{1+2}^k}\right)\right\}\cdot\exp\{H_{1+2}^k\otimes\sigma_{1+2}^k\}\right).\end{array}\tag{7}$$

Obviously the additional twistings by $\Phi_{\mathcal{J}_k}^G$ cannot be performed before the deformation of the corresponding spaces V_G^{k+1} by the extended Jordanian twists $\Phi_{\mathcal{E}_k}\Phi_{\mathcal{J}_k}$.

3 Applications

The previous result means that we have constructed explicit quantizations

$$\mathcal{R}_{\mathcal{G}_{0\prec p}}=\left(\mathcal{F}_{\mathcal{G}_{0\prec p}}\right)_{21}\left(\mathcal{F}_{\mathcal{G}_{0\prec p}}\right)^{-1}$$

of the following set of classical r-matrices:

$$r_{\mathcal{G}_{0\prec p}} = \sum_{k=0}^{p} \eta_k \left(H^k_{1+2} \wedge E^k_{1+2} + \xi_k H^k_{1-2} \wedge E^k_{1-2} + I^{1\wedge 2}_{M-4k} \right)$$

Here all the parameters are independent.

The dimensions of the nilpotent subalgebras $N^+ (so\,(M))$ in the sequence $g^\perp_{\lambda_0^p} \subset g^\perp_{\lambda_0^{p-1}} \subset \ldots \subset g^\perp_{\lambda_0^0} \subset g$ are subject to the simple relation:

$$\dim \left(N^+ \left(so\,(M) \right) \right) - \dim \left(N^+ \left(so\,(M-4) \right) \right) = 2 \left(\dim d^v_{so(M-4)} + 1 \right).$$

Taking this into account we see that the chains (7) are full in the sense that for $p = p^{\max} = [M/4] + [(M+1)/4]$ their carrier spaces contain all the generators of $N^+ (so\,(M))$. When M is even-even or odd the total number of Jordanian twists in a maximal full chain $\mathcal{F}_{\mathcal{G}_{0\prec p\max}}$ is equal to the rank of $so\,(M)$. Thus in the latter case the carrier subalgebra is equal to $B^+ (so\,(M))$.

It was demonstrated in [9] how to construct new Yangians using the explicit form of the twisting element. These new Yangians are defined by the corresponding rational solution of the matrix quantum Yang-Baxter equation (YBE). In particular, for the orthogonal classical Lie algebras $so\,(M)$ one needs the twisting element $\mathcal{F}$ in the defining (vector) representation d^v and the auxiliary operators: the flip $P : v \otimes w \to w \otimes v$ ($P \in \mathrm{Mat}\,(M) \otimes \mathrm{Mat}\,(M)$) and the operator K, which is obtained from P by transposing its first tensor factor. The following expression gives the corresponding *deformed rational solution* of the YBE:

$$u d^v \left(\mathcal{F}_{21} \mathcal{F}^{-1} \right) + P - \frac{u}{u - 1 + M/2} d^v \left(\mathcal{F}_{21} \right) K d^v \left(\mathcal{F}^{-1} \right)$$

Here u is a spectral parameter. In [10] such deformed solutions were obtained in the explicit form for the canonical chains $\mathcal{F} = \mathcal{F}_{\mathcal{B}_{0\prec p}}$.

All the calculations can be reproduced for the twisting elements $\mathcal{F} = \mathcal{F}_{\mathcal{G}_{0\prec p}}$ of the full chains. This will lead to a new set of so called *deformed Yangians* [11].

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